Novelty Journals

ISSN 2394-7314

International Journal of Novel Research in Computer Science and Software Engineering
Vol. 10, Issue 2, pp: (28-33), Month: May - August 2023, Available at: www.noveltyjournals.com

A Study on a Nonlinear Fractional Differential Equation

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DOI: https://doi.org/10.5281/zenodo.8050316

Published Date: 17-June-2023

***Abstract*: In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study a nonlinear second order fractional differential equation. The general solution of this nonlinear second order fractional differential equation can be obtained by using some techniques. Moreover, our result is a generalization of the result of ordinary differential equation.**

***Keywords*: Jumarie's modified R-L fractional calculus, new multiplication, fractional analytic functions, nonlinear second order fractional differential equation, general solution.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis, involving the research and applications of arbitrary order integrals and derivatives. Fractional calculus originated from a problem put forward by L'Hospital and Leibniz in 1695. Therefore, the history of fractional calculus was formed more than 300 years ago, and fractional calculus and classical calculus have almost the same long history. Since then, fractional calculus has attracted the attention of many contemporary great mathematicians, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl. With the efforts of researchers, the theory of fractional calculus and its applications have developed rapidly. On the other hand, fractional calculus has wide applications in physics, mechanics, electrical engineering, viscoelasticity, biology, control theory, dynamics, economics, and other fields [1-16].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [17-21]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study the following nonlinear second order α-fractional differential equation:

$$\left({}_0D_x^{\alpha}\right)[y_\alpha(x^\alpha)]\otimes_\alpha\left({}_0D_x^{\alpha}\right)^2[y_\alpha(x^\alpha)]=\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left({}_0D_x^{\alpha}\right)^2[y_\alpha(x^\alpha)]. \tag{1}$$

Where $0<\alpha\leq 1$. Using some techniques, the general solution of this nonlinear second order α-fractional differential equation can be obtained. In fact, our result is a generalization of the result of ordinary differential equation.

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II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([22])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{2}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{3}$$

where $\Gamma(\ \ )$ is the gamma function. On the other hand, for any positive integer m, we define $\left({}_{x_0}D_x^{\alpha}\right)^m[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the m-th order α-fractional derivative of $f(x)$.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([23])**:** *If α, β, x_0, c are real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{5}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([24])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([25])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{6}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{7}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{8}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{9}$$

Definition 2.5 ([26])**:** If $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}, \tag{10}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{11}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(g_\alpha(x^\alpha))^{\otimes_\alpha n}, \tag{12}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{b_n}{n!}(f_\alpha(x^\alpha))^{\otimes_\alpha n}. \tag{13}$$

Definition 2.6 ([27])**:** Let $0 < \alpha \le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)}x^\alpha. \tag{14}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([28])**:** If $0 < \alpha \le 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{15}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{16}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{17}$$

Definition 2.8 ([29])**:** Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $(f_\alpha(x^\alpha))^{\otimes_\alpha m} = f_\alpha(x^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the mth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $(f_\alpha(x^\alpha))^{\otimes_\alpha (-1)}$.

Definition 2.8 ([30])**:** Let $0 < \alpha \le 1$ and r be a real number. The r-th power of the α-fractional analytic function $f_\alpha(x^\alpha)$ is defined by

$$[f_\alpha(x^\alpha)]^{\otimes_\alpha r} = E_\alpha\left(r \cdot Ln_\alpha(f_\alpha(x^\alpha))\right). \tag{18}$$

III. MAIN RESULT

In this section, we solve a nonlinear second order fractional differential equation. At first, we need a lemma.

Lemma 3.1: *If* $0 < \alpha \le 1$, c_1 *is a constant, and* $c_1 > 0$, *then*

$$\left({}_0I_x^\alpha\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + c_1\right]^{\otimes_\alpha \left(\frac{1}{2}\right)}\right]$$

$$= \frac{c_1}{2}\left\{arcsinh_\alpha\left(\frac{1}{\sqrt{c_1}}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right) + \frac{1}{c_1}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + c_1\right)\right]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right\}. \tag{19}$$

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Proof Let $\frac{1}{\Gamma(\alpha+1)}x^{\alpha}=\sqrt{c_1}\cdot sinh_{\alpha}(\phi^{\alpha})$, then

$$\left({}_0I_x^{\alpha}\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+c_1\right]^{\otimes_{\alpha}\left(\frac{1}{2}\right)}\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+c_1\right]^{\otimes_{\alpha}\left(\frac{1}{2}\right)}\otimes_{\alpha}\left({}_0D_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]\right]$$

$$=\left({}_0I_{\phi}^{\alpha}\right)\left[\left[c_1\left([sinh_{\alpha}(\phi^{\alpha})]^{\otimes_{\alpha}2}+1\right)\right]^{\otimes_{\alpha}\left(\frac{1}{2}\right)}\otimes_{\alpha}\left[\sqrt{c_1}\cdot cosh_{\alpha}(\phi^{\alpha})\right]\right]$$

$$=\left({}_0I_{\phi}^{\alpha}\right)\left[c_1\cdot[cosh_{\alpha}(\phi^{\alpha})]^{\otimes_{\alpha}2}\right]$$

$$=\frac{c_1}{2}\left({}_0I_{\phi}^{\alpha}\right)[1+cosh_{\alpha}(2\phi^{\alpha})]$$

$$=\frac{c_1}{2}\left[\frac{1}{\Gamma(\alpha+1)}\phi^{\alpha}+\frac{1}{2}sinh_{\alpha}(2\phi^{\alpha})\right]$$

$$=\frac{c_1}{2}\left[\frac{1}{\Gamma(\alpha+1)}\phi^{\alpha}+sinh_{\alpha}(\phi^{\alpha})\otimes_{\alpha}cosh_{\alpha}(\phi^{\alpha})\right]$$

$$=\frac{c_1}{2}\left\{arcsinh_{\alpha}\left(\frac{1}{\sqrt{c_1}}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)+\frac{1}{\sqrt{c_1}}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}\left[\left(1+\frac{1}{c_1}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}\right)\right]^{\otimes_{\alpha}\left(\frac{1}{2}\right)}\right\}$$

$$=\frac{c_1}{2}\left\{arcsinh_{\alpha}\left(\frac{1}{\sqrt{c_1}}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)+\frac{1}{c_1}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+c_1\right)\right]^{\otimes_{\alpha}\left(\frac{1}{2}\right)}\right\}.$$

Q.e.d.

Theorem 3.2: *Let* $0<\alpha\leq 1$, *then the nonlinear second order fractional differential equation*

$$\left({}_0D_x^{\alpha}\right)[y_{\alpha}(x^{\alpha})]\otimes_{\alpha}\left({}_0D_x^{\alpha}\right)^2[y_{\alpha}(x^{\alpha})]=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\left({}_0D_x^{\alpha}\right)^2[y_{\alpha}(x^{\alpha})]$$

has a general solution

$$y_{\alpha}(x^{\alpha})=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\pm\frac{c_1}{2}\left\{arcsinh_{\alpha}\left(\frac{1}{\sqrt{c_1}}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)+\frac{1}{c_1}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}2}+c_1\right)\right]^{\otimes_{\alpha}\left(\frac{1}{2}\right)}\right\}+c_2\,. \quad (20)$$

Where c_1, c_2 *are constants and* $c_1>0$.

Proof Let $u_{\alpha}(x^{\alpha})=\left({}_0D_x^{\alpha}\right)[y_{\alpha}(x^{\alpha})]$, then

$$u_{\alpha}(x^{\alpha})\otimes_{\alpha}\left({}_0D_x^{\alpha}\right)[u_{\alpha}(x^{\alpha})]=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\left({}_0D_x^{\alpha}\right)[u_{\alpha}(x^{\alpha})]. \quad (21)$$

It follows that

$$[u_{\alpha}(x^{\alpha})-1]\otimes_{\alpha}\left({}_0D_x^{\alpha}\right)[u_{\alpha}(x^{\alpha})]=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}. \quad (22)$$

And hence,

$$\left({}_0I_{u_{\alpha}}^{\alpha}\right)[u_{\alpha}(x^{\alpha})-1]=\left({}_0I_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]. \quad (23)$$

Thus,

$$\frac{1}{2}[u_\alpha(x^\alpha)-1]^{\otimes_\alpha 2}=\frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\frac{1}{2}c_1. \tag{24}$$

Where c_1 is a constant. Therefore,

$$[u_\alpha(x^\alpha)-1]^{\otimes_\alpha 2}=\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+c_1. \tag{25}$$

Hence,

$$u_\alpha(x^\alpha)=1\pm\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+c_1\right]^{\otimes_\alpha\left(\frac{1}{2}\right)}. \tag{26}$$

That is,

$$\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)]=1\pm\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+c_1\right]^{\otimes_\alpha\left(\frac{1}{2}\right)}. \tag{27}$$

Thus,

$$y_\alpha(x^\alpha)=\frac{1}{\Gamma(\alpha+1)}x^\alpha\pm\left({}_0I_x^\alpha\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+c_1\right]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right]+c_2. \tag{28}$$

Therefore, by Lemma 3.1, we have

$$y_\alpha(x^\alpha)=\frac{1}{\Gamma(\alpha+1)}x^\alpha\pm\frac{c_1}{2}\left\{arcsinh_\alpha\left(\frac{1}{\sqrt{c_1}}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)+\frac{1}{c_1}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+c_1\right)\right]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right\}+c_2.$$

Where c_1, c_2 are constants and $c_1>0$. Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we study a nonlinear second order fractional differential equation. The general solution of this nonlinear second order fractional differential equation can be obtained by using some methods. In addition, our result is a generalization of the result of ordinary differential equation. In the future, we will continue to use Jumarie's modified R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in applied mathematics and fractional differential equations.

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